连续小波变换理论

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- 一维连续小波变换
- 各向同性小波变换
- 各向异性小波变换
- 小波变换的离散化

一维连续小波变换

什么是小波

- 小波(wavelet, $\psi(x)$)：有限长或迅速衰减的振荡函数

容许性

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(k)|^2}{k} \mathrm{d}k < \infty$$

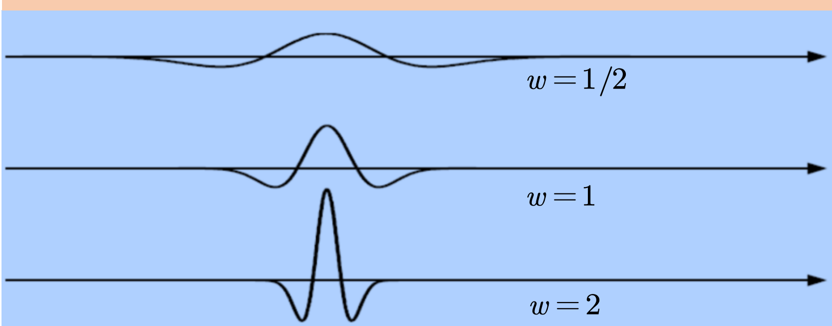
平方可积性

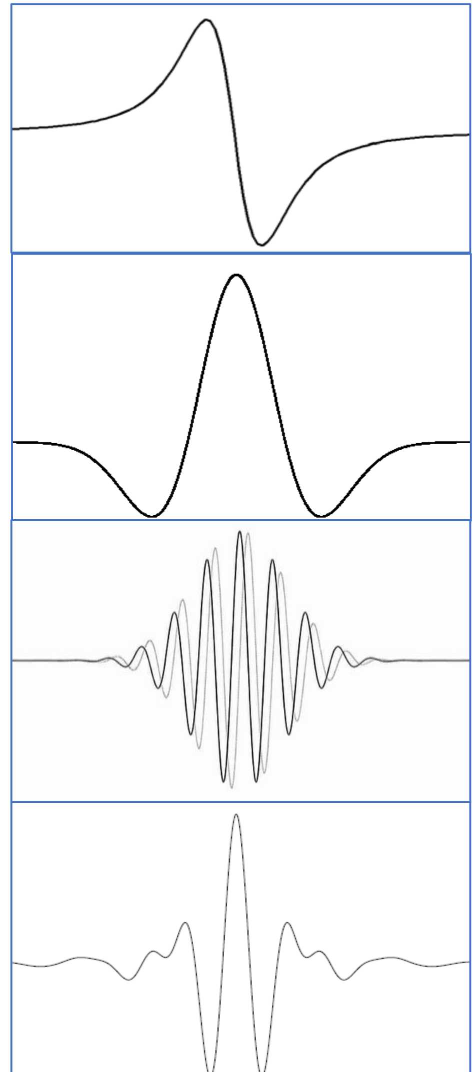
$$E_\psi = \int_{-\infty}^{+\infty} |\psi(x)|^2 \mathrm{d}x < \infty$$

通常将其归一化

平移性 $\psi(x-u)$

伸缩性 $\sqrt{w}\psi(wx)$， $w>0$





1st order derivative of Gaussian wavelet

cubic B-spline wavelet, Mexican hat wavelet, Gaussian-derived wavelet

Morlet wavelet

Meyer wavelet

一维连续小波变换

连续小波变换

连续小波变换
Continuous wavelet transform (CWT)

$$W_f(w,x)=\int_{-\infty}^{+\infty} f(u)\sqrt{w}\,\psi[w(x-u)]\mathrm{d}u$$

$$=\int_{-\infty}^{+\infty} f(u)\psi(w,x-u)\mathrm{d}u$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k)\hat{\psi}(w,k)e^{-ikx}\mathrm{d}k$$

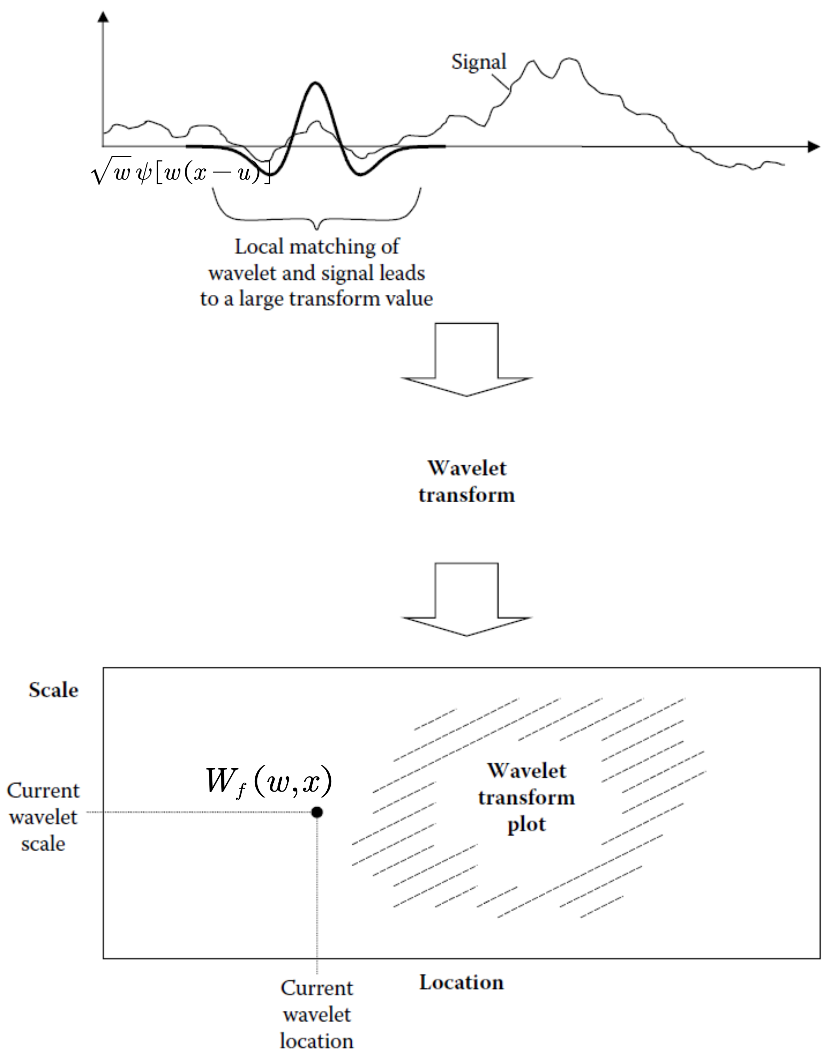
$\hat{W}_f(w,k)$

子小波

$$\psi(w,x)=\sqrt{w}\,\psi(wx)$$

子小波的Fourier 变换

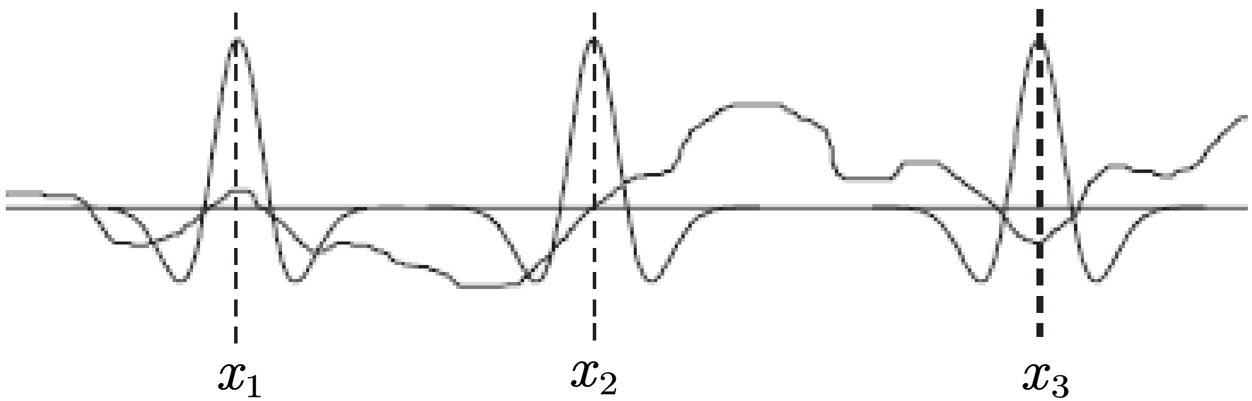
$$\hat{\psi}(w,k)=\frac{1}{\sqrt{w}}\hat{\psi}(k/w)$$



一维连续小波变换

如何理解$W_f(w,x)$

- 给定尺度w

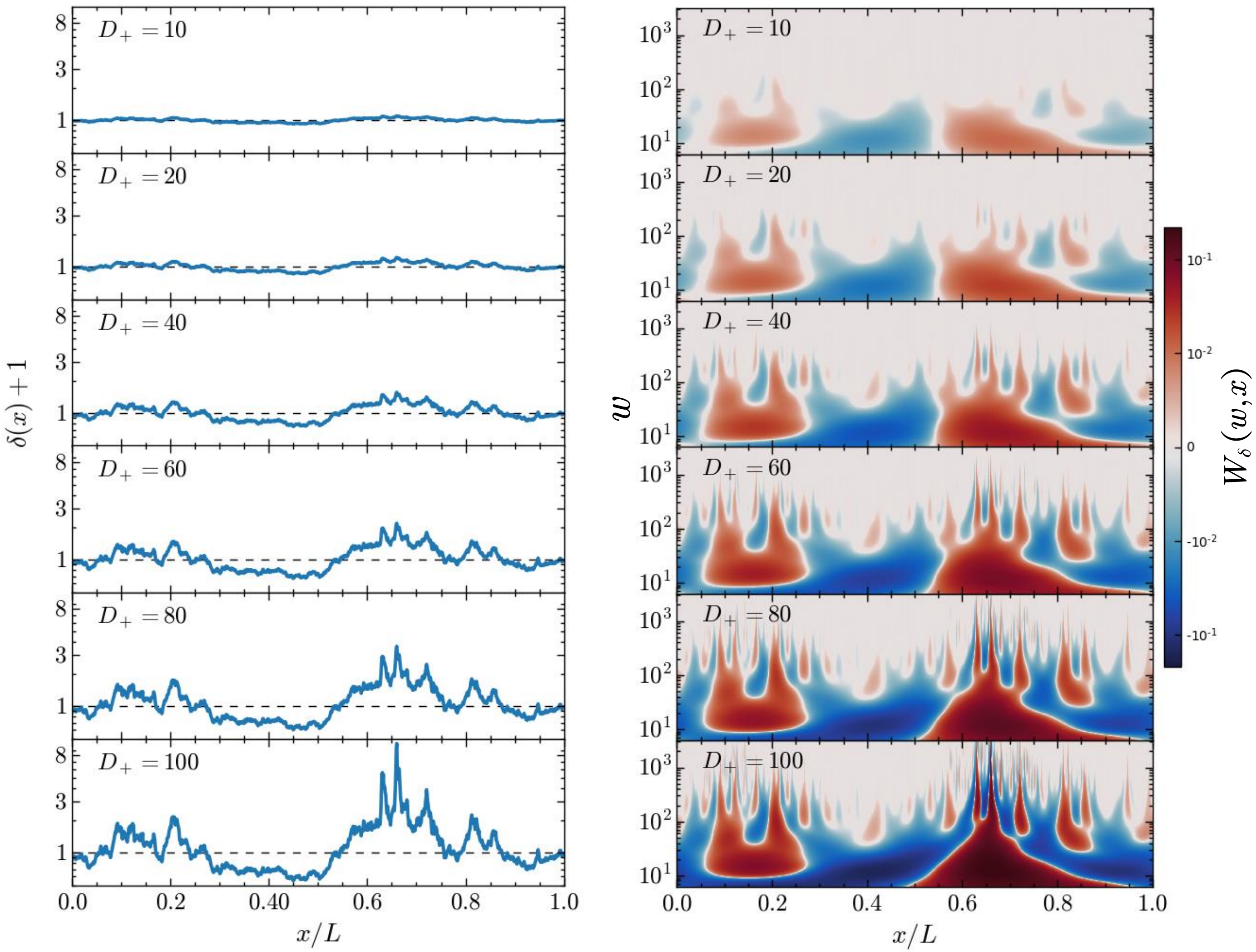


$W_f(w,x_1)>0$ $W_f(w,x_2)=0$ $W_f(w,x_3)<0$

一维连续小波变换

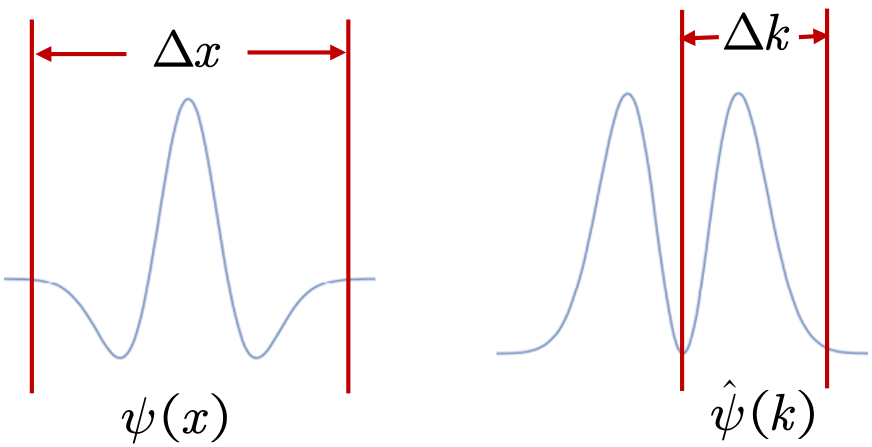
如何理解$W_f(w,x)$

- 例

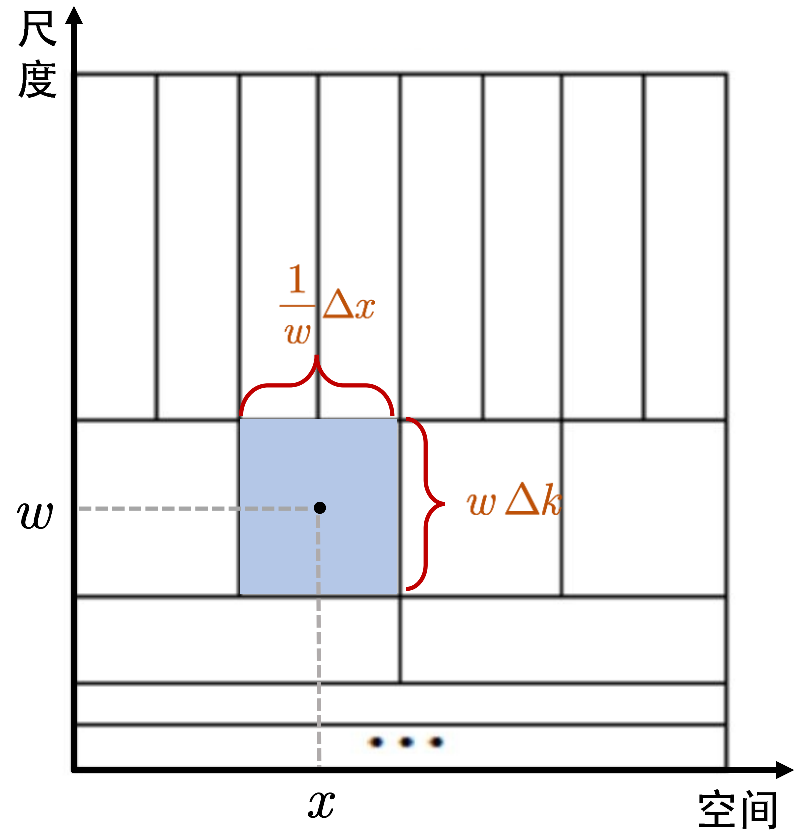


一维连续小波变换

空间-尺度分辨率



- $\psi(w,x)$的宽度：$\frac{1}{w}\Delta x$
- $\hat{\psi}(w,k)$的宽度：$w\Delta k$



$$\frac{1}{w}\Delta x \cdot w\,\Delta k = \Delta x \cdot \Delta k = \text{Constant}$$

一维连续小波变换

空间-尺度分辨率

- 在$k>0$的正半轴上，$\hat{\psi}(k)$的中心为

$$k_c=\frac{\int_0^{\infty}k|\hat{\psi}(k)|^2\mathrm{d}k}{\int_0^{\infty}|\hat{\psi}(k)|^2\mathrm{d}k}$$

- 在$k>0$的正半轴上，$\hat{\psi}(k)$的宽度为

$$\Delta k=2\left(\frac{\int_0^{\infty}(k-k_c)^2|\hat{\psi}(k)|^2\mathrm{d}k}{\int_0^{\infty}|\hat{\psi}(k)|^2\mathrm{d}k}\right)^{1/2}$$

- $\psi(x)$的宽度为

$$\Delta x=2\left(\frac{\int_{-\infty}^{\infty}x^2|\psi(x)|^2\mathrm{d}x}{\int_{-\infty}^{\infty}|\psi(x)|^2\mathrm{d}x}\right)^{1/2}$$

(Chui 1997)

- 高斯诱导小波(Gaussian-derived wavelet, GDW):

$$\psi(x)=\frac{1}{(18\pi)^{1/4}}(2-x^2)e^{-x^2/4}$$

$$\hat{\psi}(k)=4\left(\frac{8\pi}{9}\right)^{1/4}k^2e^{-k^2}$$

$$\Delta x\Delta k\approx 2.101$$

- 余弦加权的高斯诱导小波(cosine-weighted Gaussian-derived wavelet, CW-GDW):

$$\psi(x)=\frac{2}{\pi^{1/4}}\sqrt{\frac{2e}{1+5e}}[(1-x^2)\cos x-x\sin x]e^{-x^2/2}$$

$$\hat{\psi}(k)=\frac{4\pi^{1/4}}{\sqrt{1+5e}}k(k\cosh k-\sinh k)e^{-k^2/2}$$

$$\Delta x\Delta k\approx 2.035$$

一维连续小波变换

与小波尺度对应的Fourier波数/频率

- 为了比较基于不同类型小波的CWT，以及比较基于CWT和基于Fourier变换的谱分析结果，需要理清小波尺度与Fourier波数的关系
- 根据Meyers et al. 1993和Torrence et al. 1998，与小波尺度w相对应的波数k_{pseu}可以通过计算$\cos(k_{\text{pseu}}\, x)$的CWT模方的最大值来确定

$$\cos(k_{\text{pseu}}x) \xrightarrow{\text{CWT}} W_{\cos}(w,x) = \cos(k_{\text{pseu}}x)\,\hat{\psi}(w,k_{\text{pseu}})$$

模方 ↓

$$|W_{\cos}(w,x)|^2 = \cos^2(k_{\text{pseu}}x)|\hat{\psi}(w,k_{\text{pseu}})|^2$$

- $|W_{\cos}(w,x)|^2$达到最大值时的w由$\dfrac{\partial |W_{\cos}(w,x)|^2}{\partial w} = 0$决定

||

$$\frac{\partial |\hat{\psi}(w,k_{\text{pseu}})|^2}{\partial w} = 0$$

与小波尺度对应的Fourier波数/频率

$$\frac{\partial|\hat{\psi}(w,k_{\text{pseu}})|^2}{\partial w}=0$$

$\hat{\psi}(w,k)=\frac{1}{\sqrt{w}}\hat{\psi}(k/w)$

$$\frac{\partial|\hat{\psi}(k_{\text{pseu}}/w)|^2/w}{\partial w}=0$$

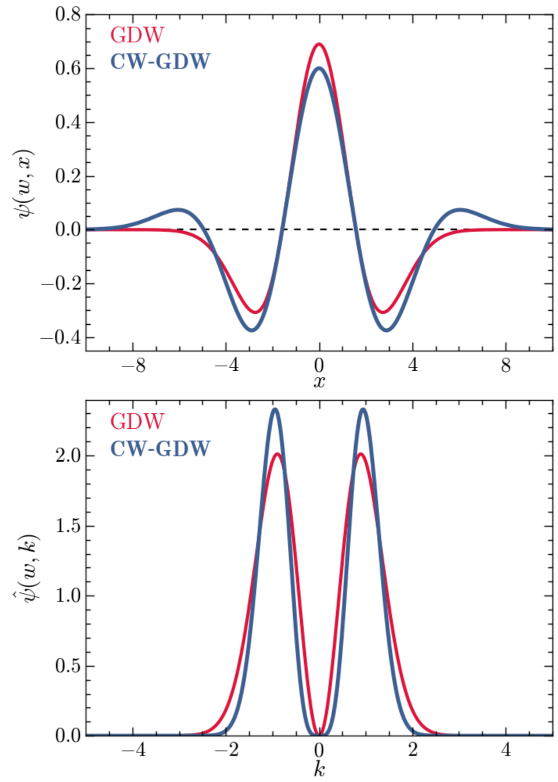
令$\frac{1}{c_w}=\frac{k_{\text{pseu}}}{w}$

$$\frac{\partial|\hat{\psi}(1/c_w)|^2/c_w}{\partial c_w}=0$$

$$w=c_w k_{\text{pseu}}$$

- GDW：$c_w \approx 0.8944$
- CW-GDW：$c_w \approx 0.4282$

- 给定尺度$w=c_w$
  - GDW：$\Delta k \approx 0.6151$，$\Delta x \approx 3.4158$
  - CW-GDW：$\Delta k \approx 0.4586$，$\Delta x \approx 4.4372$



一维连续小波变换

传统的连续小波逆变换

$$\hat{W}_f(w,k) = \hat{f}(k)\hat{\psi}(w,k)$$

等号两端乘以$\hat{\psi}^*(w,k)$

$$\hat{W}_f(w,k)\hat{\psi}^*(w,k) = \hat{f}(k)|\hat{\psi}(w,k)|^2$$

Fourier逆变换

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{W}_f(w,k)\hat{\psi}^*(w,k)e^{-ikx}\mathrm{d}k = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)|\hat{\psi}(w,k)|^2 e^{-ikx}\mathrm{d}k$$

$$\int_{-\infty}^{+\infty}W_f(w,u)\psi^*(w,x-u)\mathrm{d}u = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)e^{-ikx}\frac{|\hat{\psi}(k/w)|^2}{w}\mathrm{d}k$$

在等式两端对w积分

$$\int_0^{\infty}\left(\int_{-\infty}^{+\infty}W_f(w,u)\psi^*(w,x-u)\mathrm{d}u\right)\mathrm{d}w = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)e^{-ikx}\left(\int_0^{+\infty}\frac{|\hat{\psi}(k/w)|^2}{w}\mathrm{d}w\right)\mathrm{d}k$$

一维连续小波变换

传统的连续小波逆变换

$$\int_0^{\infty}\left(\int_{-\infty}^{+\infty} W_f(w,u)\psi^*(w,x-u)\mathrm{d}u\right)\mathrm{d}w=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)e^{-ikx}\left(\int_0^{+\infty}\frac{|\hat{\psi}(k/w)|^2}{w}\mathrm{d}w\right)\mathrm{d}k$$

$\psi(x)$为复数小波，且其Fourier变换满足$\hat{\psi}(k<0)=0$

- $k=0$：$\int_0^{+\infty}\frac{|\hat{\psi}\left(\frac{k}{w}\right)|^2}{w}\mathrm{d}w=0$；
- $k>0$：令$k'=k/w$，则$\int_0^{+\infty}\frac{|\hat{\psi}\left(\frac{k}{w}\right)|^2}{w}\mathrm{d}w=\int_0^{+\infty}\frac{|\hat{\psi}(k')|^2}{k'}\mathrm{d}k'$；
- $k<0$：令$k'=-k/w$，则$\int_0^{+\infty}\frac{|\hat{\psi}\left(\frac{k}{w}\right)|^2}{w}\mathrm{d}w=\int_0^{+\infty}\frac{|\hat{\psi}(-k')|^2}{k'}\mathrm{d}k'=0$

定义$C_\psi=\int_0^{+\infty}\frac{|\hat{\psi}(k)|^2}{k}\mathrm{d}k$，若$0<C_\psi<\infty$，则有

$$\frac{1}{C_\psi}\int_0^{\infty}\left(\int_{-\infty}^{+\infty} W_f(w,u)\psi^*(w,x-u)\mathrm{d}u\right)\mathrm{d}w=\frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k$$

一维连续小波变换

传统的连续小波逆变换

定义$C_\psi = \int_0^{+\infty} \frac{|\widehat{\psi}(k)|^2}{k} \mathrm{d}k$，若$0 < C_\psi < \infty$，则有

$$\frac{1}{C_\psi}\int_0^{\infty}\left(\int_{-\infty}^{+\infty} W_f(w,u)\psi^*(w,x-u)\mathrm{d}u\right)\mathrm{d}w = \frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k$$

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)e^{-ikx}\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{k<0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \lim_{\delta k\to 0}\frac{\delta k}{2\pi}\hat{f}(0) \\ &= \left(\frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k\right)^* + \frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \lim_{L\to\infty}\frac{1}{L}\int_{-L/2}^{L/2}f(x)\mathrm{d}x \\ &= 2\mathrm{Re}\left\{\frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k\right\} + \overline{f} \end{aligned}$$

$$f(x) = \overline{f} + \frac{2}{C_\psi}\mathrm{Re}\left\{\int_0^{\infty}\left(\int_{-\infty}^{+\infty} W_f(w,u)\psi^*(w,x-u)\mathrm{d}u\right)\mathrm{d}w\right\}$$

一维连续小波变换

传统的连续小波逆变换

$$\int_0^{\infty}\left(\int_{-\infty}^{+\infty}W_f(w,u)\psi^*(w,x-u)\mathrm{d}u\right)\mathrm{d}w=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)e^{-ikx}\left(\int_0^{+\infty}\frac{|\hat{\psi}(k/w)|^2}{w}\mathrm{d}w\right)\mathrm{d}k$$

$\psi(x)$为实数小波，则其Fourier变换满足$\hat{\psi}(-k)=\hat{\psi}^*(k)$

- $k=0$： $\int_0^{+\infty}\frac{|\hat{\psi}\left(\frac{k}{w}\right)|^2}{w}\mathrm{d}w=0$；
- $k>0$：令$k'=k/w$，则$\int_0^{+\infty}\frac{|\hat{\psi}\left(\frac{k}{w}\right)|^2}{w}\mathrm{d}w=\int_0^{+\infty}\frac{|\hat{\psi}(k')|^2}{k'}\mathrm{d}k'$；
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$$\frac{1}{C_\psi}\int_0^{\infty}\left(\int_{-\infty}^{+\infty}W_f(w,u)\psi(w,x-u)\mathrm{d}u\right)\mathrm{d}w=\frac{1}{2\pi}\int_{k\neq 0}\hat{f}(k)e^{-ikx}\mathrm{d}k$$

一维连续小波变换

传统的连续小波逆变换

定义$C_\psi = \int_0^{+\infty} \frac{|\widehat{\psi}(k)|^2}{k} \mathrm{d}k$，若$0 < C_\psi < \infty$，则有

$$\frac{1}{C_\psi}\int_0^{\infty}\left(\int_{-\infty}^{+\infty} W_f(w,u)\psi(w,x-u)\mathrm{d}u\right)\mathrm{d}w = \frac{1}{2\pi}\int_{k\neq 0}\hat{f}(k)e^{-ikx}\mathrm{d}k$$

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)e^{-ikx}\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{k\neq 0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \lim_{\delta k\to 0}\frac{\delta k}{2\pi}\hat{f}(0) \\ &= \frac{1}{2\pi}\int_{k\neq 0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \overline{f} \end{aligned}$$

$$f(x) = \overline{f} + \frac{1}{C_\psi}\int_0^{\infty}\left(\int_{-\infty}^{+\infty} W_f(w,u)\psi(w,x-u)\mathrm{d}u\right)\mathrm{d}w$$

一维连续小波变换

传统的连续小波逆变换

满足$\hat{\psi}(k<0)=0$的复数小波

$$f(x)=\overline{f}+\frac{2}{C_\psi}\mathrm{Re}\left\{\int_0^\infty\left(\int_{-\infty}^{+\infty}W_f(w,u)\psi^*(w,x-u)\mathrm{d}u\right)\mathrm{d}w\right\}$$

实数小波

$$f(x)=\overline{f}+\frac{1}{C_\psi}\int_0^\infty\left(\int_{-\infty}^{+\infty}W_f(w,u)\psi(w,x-u)\mathrm{d}u\right)\mathrm{d}w$$

一维连续小波变换

新型连续小波逆变换

$$W_f(w,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)\frac{\hat{\psi}(k/w)}{\sqrt{w}}e^{-ikx}\mathrm{d}k$$

等号两端除以$\sqrt{w}$并对w积分

$$\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)\left(\int_0^{+\infty}\frac{\hat{\psi}(k/w)}{w}\mathrm{d}w\right)e^{-ikx}\mathrm{d}k$$

$\psi(x)$为复数小波，且其Fourier变换满足$\hat{\psi}(k<0)=0$

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- $k<0$：令$k'=-k/w$，则$\int_0^{+\infty}\frac{\hat{\psi}\left(\frac{k}{w}\right)}{w}\mathrm{d}w=\int_0^{+\infty}\frac{\hat{\psi}(-k')}{k'}\mathrm{d}k'=0$

定义$K_\psi=\int_0^{+\infty}\frac{\hat{\psi}(k)}{k}\mathrm{d}k$，若$0<|K_\psi|<\infty$，则有

$$\frac{1}{K_\psi}\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w=\frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k$$

一维连续小波变换

新型连续小波逆变换

定义$K_\psi = \int_0^{+\infty} \frac{\widehat{\psi}(k)}{k} \mathrm{d}k$，若$0 < |K_\psi| < \infty$，则有

$$\frac{1}{K_\psi}\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w = \frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k$$

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)e^{-ikx}\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{k<0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \lim_{\delta k\to 0}\frac{\delta k}{2\pi}\hat{f}(0) \\ &= \left(\frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k\right)^* + \frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k + \lim_{L\to\infty}\frac{1}{L}\int_{-L/2}^{L/2}f(x)\mathrm{d}x \\ &= 2\mathrm{Re}\left\{\frac{1}{2\pi}\int_{k>0}\hat{f}(k)e^{-ikx}\mathrm{d}k\right\} + \overline{f} \end{aligned}$$

$$f(x) = \overline{f} + 2\mathrm{Re}\left\{\frac{1}{K_\psi}\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w\right\}$$

一维连续小波变换

新型连续小波逆变换

$$W_f(w,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)\frac{\hat{\psi}(k/w)}{\sqrt{w}}e^{-ikx}\mathrm{d}k$$

等号两端除以$\sqrt{w}$并对w积分

$$\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)\left(\int_0^{+\infty}\frac{\hat{\psi}(k/w)}{w}\mathrm{d}w\right)e^{-ikx}\mathrm{d}k$$

$\psi(x)$为实对称小波，则其Fourier变换满足$\hat{\psi}(k)=\hat{\psi}(-k)$

- $k=0$： $\int_0^{+\infty}\frac{\hat{\psi}\left(\frac{k}{w}\right)}{w}\mathrm{d}w=0$；
- $k>0$：令$k'=k/w$，则$\int_0^{+\infty}\frac{\hat{\psi}\left(\frac{k}{w}\right)}{w}\mathrm{d}w=\int_0^{+\infty}\frac{\hat{\psi}(k')}{k'}\mathrm{d}k'$；
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定义$K_\psi=\int_0^{+\infty}\frac{\hat{\psi}(k)}{k}\mathrm{d}k$，若$0<|K_\psi|<\infty$，则有

$$\frac{1}{K_\psi}\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w=\frac{1}{2\pi}\int_{k\neq 0}\hat{f}(k)e^{-ikx}\mathrm{d}k$$

新型连续小波逆变换

定义$K_\psi = \int_0^{+\infty} \frac{\widehat{\psi}(k)}{k} \mathrm{d}k$，若$0 < |K_\psi| < \infty$，则有

$$\frac{1}{\mathcal{K}_\psi}\int_0^{+\infty} \frac{W_f(w,x)}{\sqrt{w}} \mathrm{d}w = \frac{1}{2\pi}\int_{k\neq 0} \hat{f}(k) e^{-ikx} \mathrm{d}k$$

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k) e^{-ikx} \mathrm{d}k \\ &= \frac{1}{2\pi}\int_{k\neq 0} \hat{f}(k) e^{-ikx} \mathrm{d}k + \lim_{\delta k \to 0} \frac{\delta k}{2\pi} \hat{f}(0) \\ &= \frac{1}{2\pi}\int_{k\neq 0} \hat{f}(k) e^{-ikx} \mathrm{d}k + \overline{f} \end{aligned}$$

$$f(x) = \overline{f} + \frac{1}{\mathcal{K}_\psi}\int_0^{+\infty} \frac{W_f(w,x)}{\sqrt{w}} \mathrm{d}w$$

一维连续小波变换

新型连续小波逆变换

满足$\hat{\psi}(k<0)=0$的复数小波

$$f(x)=\bar{f}+2\mathrm{Re}\left\{\frac{1}{\mathcal{K}_\psi}\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w\right\}$$

Morlet公式，Daubechies et al. 2011

实对称小波

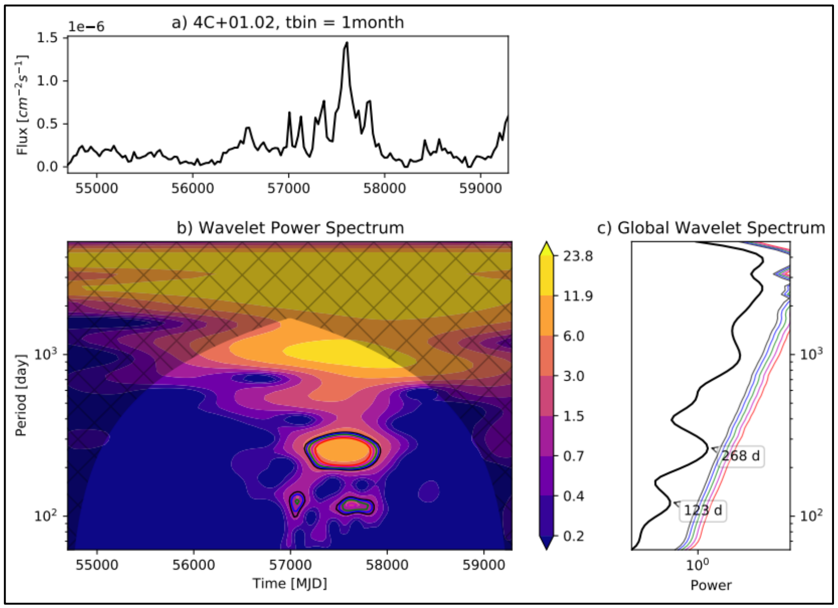
$$f(x)=\bar{f}+\frac{1}{\mathcal{K}_\psi}\int_0^{+\infty}\frac{W_f(w,x)}{\sqrt{w}}\mathrm{d}w$$

一维连续小波变换

一维连续小波变换的应用

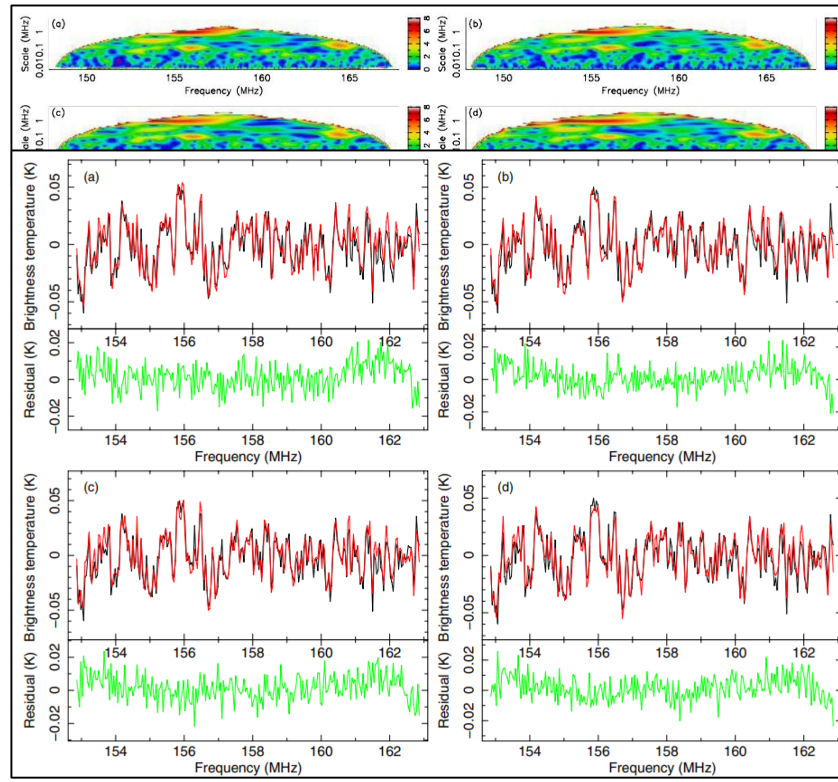
分析天体的光变曲线

Tarnopolski et al. 2020, Ren et al. 2022



去除21cm 信号的前景发射

Gu et al. 2013, Li et al. 2022

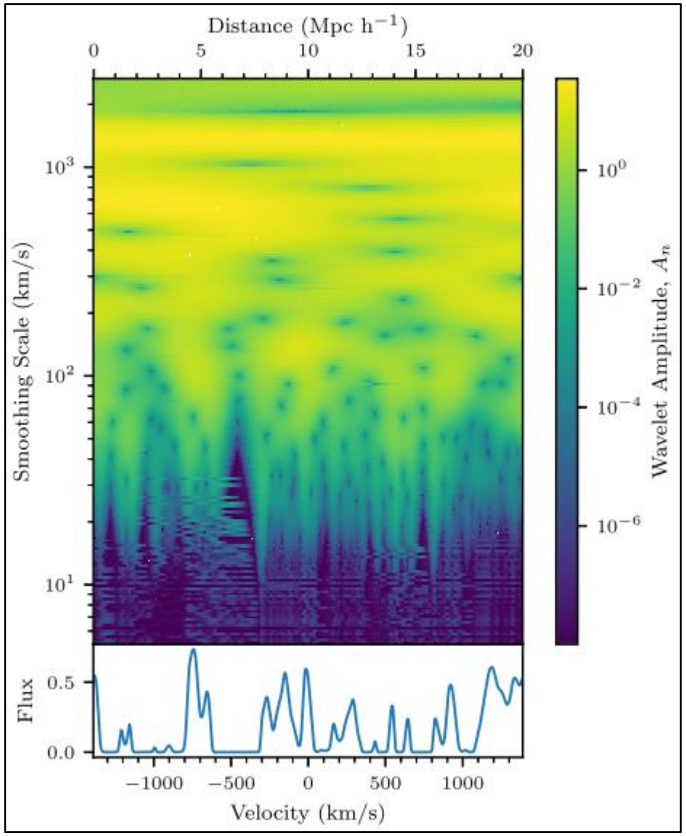


一维连续小波变换

一维连续小波变换的应用

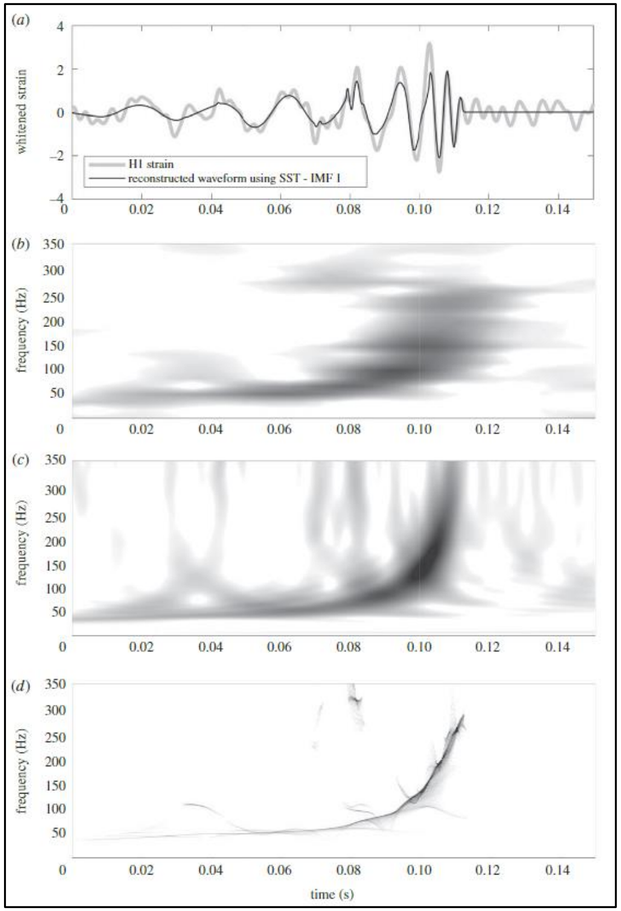
测量Lyman-α 森林的小尺度结构

Wolfson et al. 2021



分析引力波的时频特性

Tary et al. 2018



各向同性小波变换

各向同性小波

$$\Psi(w,\vec{x})=\Psi(w,r)=w^{3/2}\Psi(wr)$$
$$\hat{\Psi}(w,\vec{k})=\hat{\Psi}(w,k)=w^{-3/2}\hat{\Psi}(k/w)$$

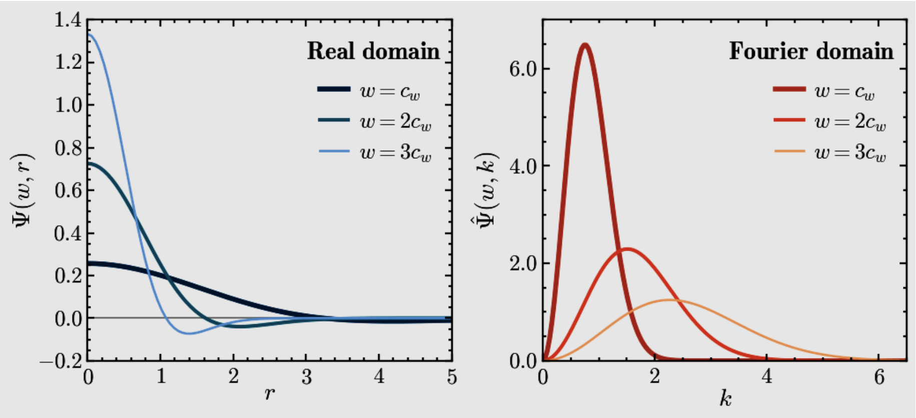
$$r=|\vec{x}| \qquad \iiint_{-\infty}^{+\infty}|\Psi(w,\vec{x})|^2\mathrm{d}^3\vec{x}=1$$
$$k=|\vec{k}| \qquad \frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty}|\hat{\Psi}(w,\vec{k})|^2\mathrm{d}^3\vec{k}=1$$

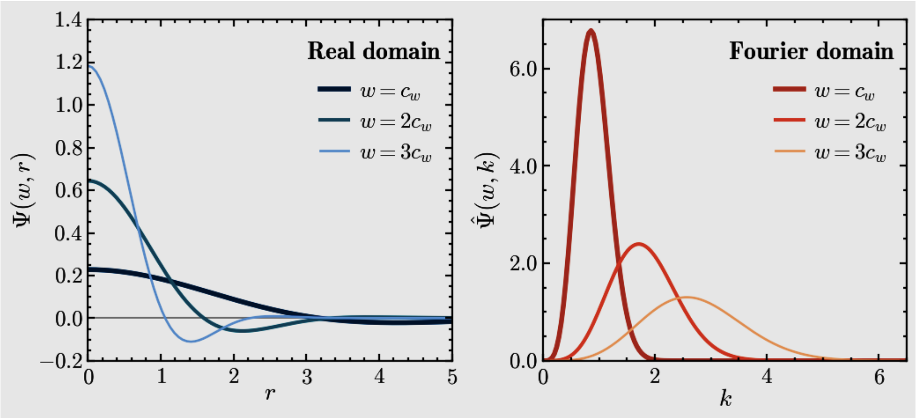
各向同性GDW

$$\Psi(r)=\frac{1}{(2\pi)^{3/4}\sqrt{15}}(6-r^2)e^{-r^2/4}$$
$$\hat{\Psi}(k)=\frac{16(2\pi^3)^{1/4}}{\sqrt{15}}k^2e^{-k^2}$$

各向同性CW-GDW

$$\Psi(r)=\frac{2}{\pi^{3/4}}\sqrt{\frac{2e}{9+55e}}\left[(4-r^2)\cos r+2\left(\frac{1}{r}-r\right)\sin r\right]e^{-r^2/2}$$
$$\hat{\Psi}(k)=\frac{8\pi^{3/4}}{\sqrt{9+55e}}k(k\cosh k-\sinh k)e^{-\frac{k^2}{2}}$$





各向同性小波变换

各向同性小波变换及其逆变换

$$W_f(w,\vec{x}) = \iiint_{-\infty}^{+\infty} f(\vec{u})\Psi(w,\vec{x}-\vec{u})\mathrm{d}^3\vec{u}$$

$$f(\vec{x}) = \overline{f} + \frac{1}{\mathcal{K}_\Psi}\int_0^{+\infty} w^{1/2} W_f(w,\vec{x})\mathrm{d}w$$

$$\overline{f} = \lim_{V\to\infty}\frac{1}{V}\int_V f(\vec{x})\mathrm{d}^3\vec{x}$$

$$0 < \left|\mathcal{K}_\Psi = \int_0^{+\infty}\frac{\hat{\Psi}(k)}{k}\mathrm{d}k\right| < \infty$$

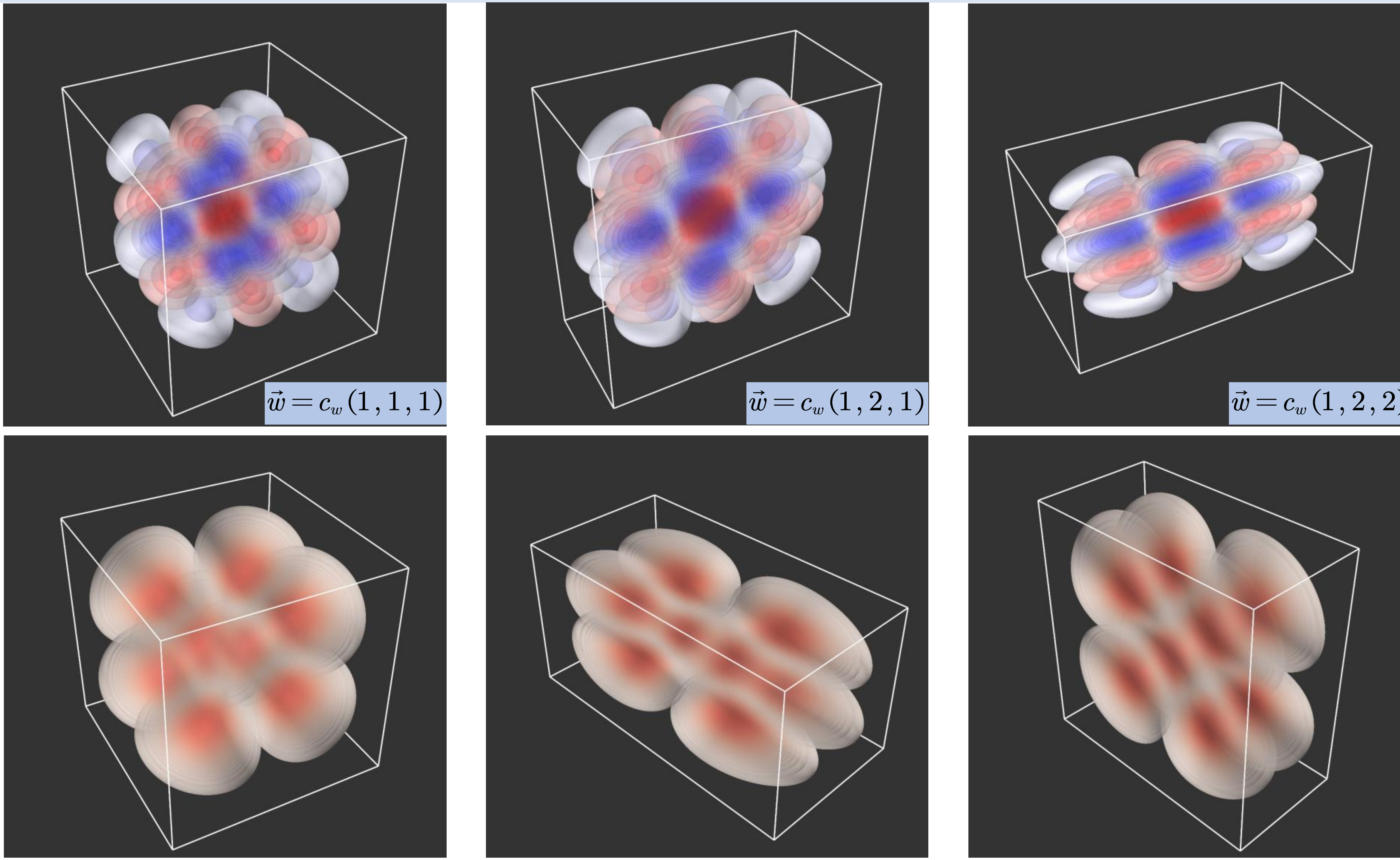
各向异性小波变换

可分离变量的各向异性小波

$$\Psi(\vec{w},\vec{x}) = \psi(w_x,x)\psi(w_y,y)\psi(w_z,z)$$
$$= (w_x w_y w_z)^{1/2}\psi(w_x x)\psi(w_y y)\psi(w_z z)$$
$$\hat{\Psi}\left(\vec{w},\vec{k}\right) = \hat{\psi}(w_x,k_x)\hat{\psi}(w_y,k_y)\hat{\psi}(w_z,k_z)$$
$$= (w_x w_y w_z)^{-1/2}\psi\left(\frac{k_x}{w_x}\right)\psi\left(\frac{k_y}{w_y}\right)\psi\left(\frac{k_z}{w_z}\right)$$

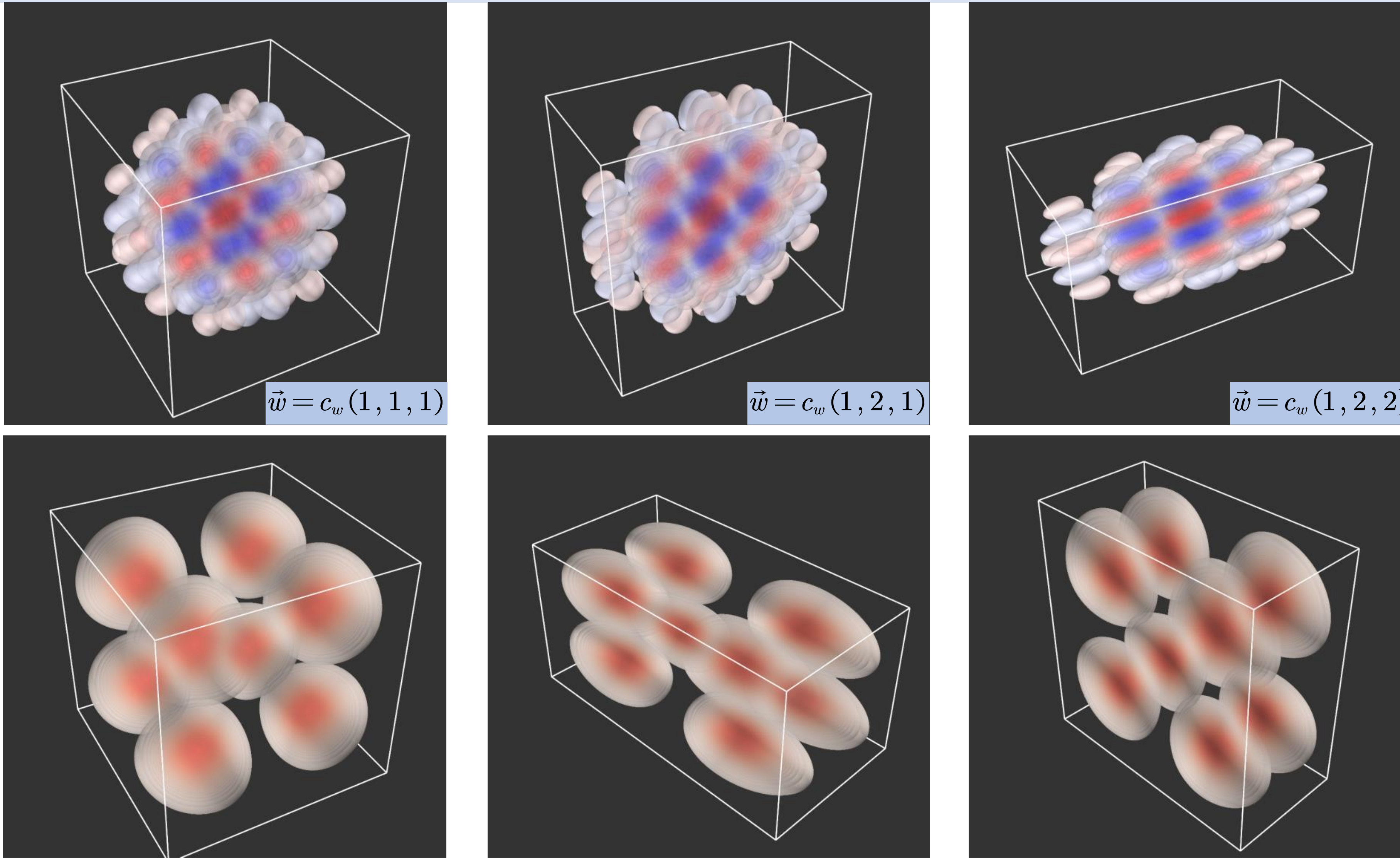
各向异性小波变换

可分离变量的各向异性GDW



各向异性小波变换

可分离变量的各向异性CW-GDW



各向异性小波变换

可分离变量的各向异性小波变换及其逆变换

$$W_f(\vec{w},\vec{x})=\iiint_{-\infty}^{+\infty} f(\vec{u})\Psi(\vec{w},\vec{x}-\vec{u})\mathrm{d}^3\vec{u}$$

$$f(\vec{x})=\bar{f}+\frac{1}{\mathcal{K}_\Psi}\iiint_{0}^{+\infty}(w_x w_y w_z)^{-1/2}W_f(\vec{w},\vec{x})\mathrm{d}^3\vec{w}$$

$$\bar{f}=\lim_{V\to\infty}\frac{1}{V}\int_V f(\vec{x})\mathrm{d}^3\vec{x}$$

$$0<\left|\mathcal{K}_\Psi=\int_0^{+\infty}\frac{\hat{\psi}(k)}{k}\mathrm{d}k\right|<\infty$$

各向异性小波变换

定向的各向异性小波

$$\Psi(w,\theta,\phi,\vec{x}) = w^{3/2}\Psi(w\mathrm{R}_{\theta\phi}\vec{x})$$
$$\hat{\Psi}\left(w,\theta,\phi,\vec{k}\right) = w^{-3/2}\hat{\Psi}\left(\mathrm{R}_{\theta\phi}\vec{k}/w\right)$$

[注]

$$\mathrm{R}_{\theta\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \cos\theta\sin\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

$$\mathrm{R}_{\theta\phi}^{-1} = \mathrm{R}_{\theta\phi}^{\mathrm{T}} = \begin{pmatrix} \cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ -\sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathrm{Det}\left(\mathrm{R}_{\theta\phi}\right) = 1 \qquad 0 \le \phi \le 2\pi, \qquad 0 \le \theta \le \pi$$

[注]：“A rotation in the spatial domain corresponds to an identical rotation in the frequency domain.” (https://see.stanford.edu/materials/lsoftaee261/book-fall-07.pdf)

各向异性小波变换

定向各向异性小波

定向的各向异性GDW

$$\Psi(\vec{x}) = \frac{1}{(2\pi)^{3/4}\sqrt{15\epsilon_1\epsilon_2}}(6-r_\epsilon^2)e^{-\frac{r_\epsilon^2}{4}}$$

$$\hat{\Psi}(\vec{k}) = 16(2\pi^3)^{1/4}\sqrt{\frac{\epsilon_1\epsilon_2}{15}}k_\epsilon^2 e^{-k_\epsilon^2}$$

定向的各向异性CW-GDW

$$\Psi(\vec{x}) = \frac{2}{\pi^{3/4}}\sqrt{\frac{2e}{(9+55e)\epsilon_1\epsilon_2}}\left[(4-r_\epsilon^2)\cos r_\epsilon + 2\left(\frac{1}{r_\epsilon}-r_\epsilon\right)\sin r_\epsilon\right]e^{-r_\epsilon^2/2}$$

$$\hat{\Psi}(\vec{k}) = 8\pi^{3/4}\sqrt{\frac{\epsilon_1\epsilon_2}{9+55e}}k_\epsilon(k_\epsilon\cosh k_\epsilon - \sinh k_\epsilon)e^{-\frac{k_\epsilon^2}{2}}$$

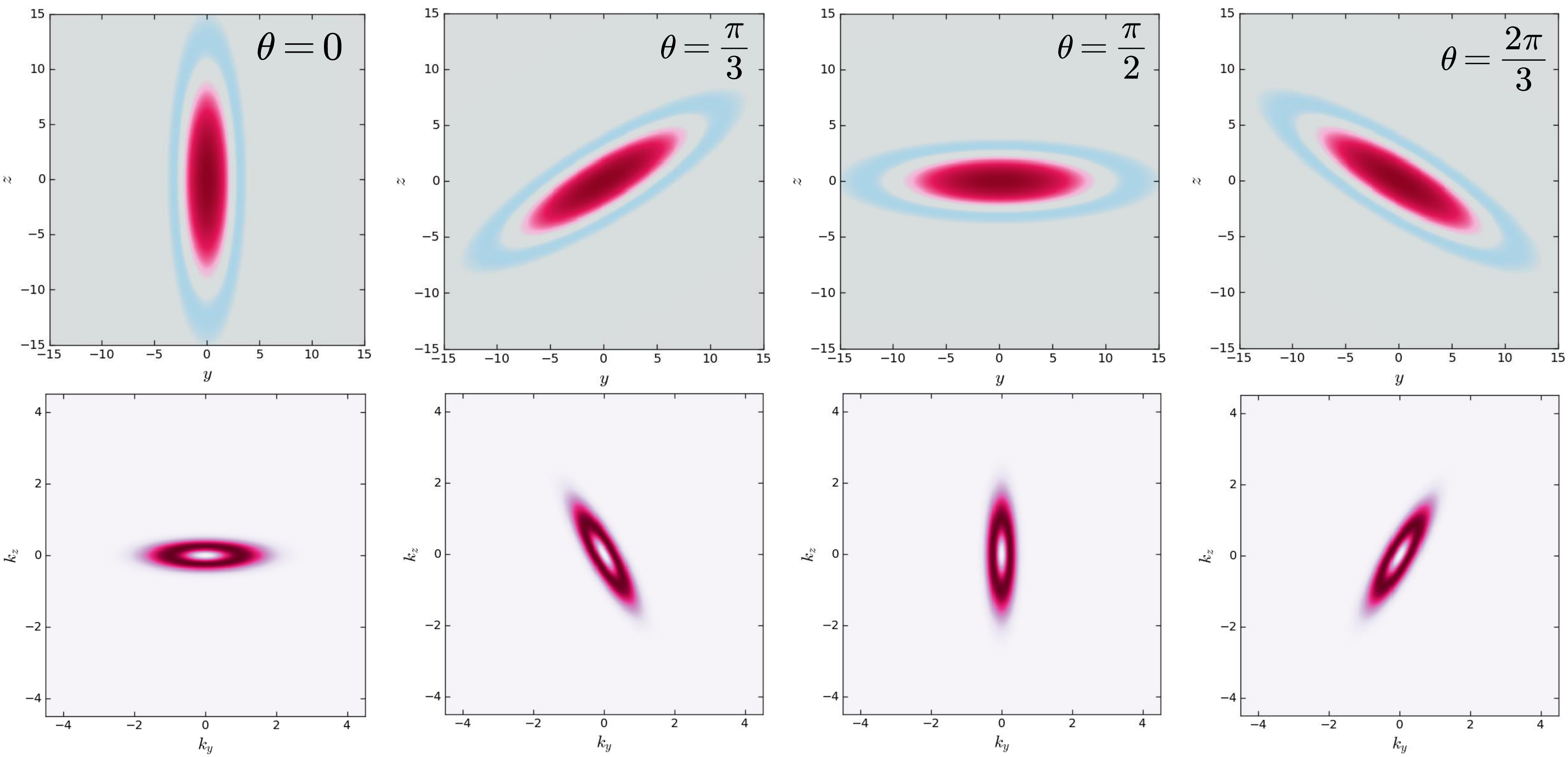
$$r_\epsilon = \sqrt{x^2+\left(\frac{y}{\epsilon_1}\right)^2+\left(\frac{z}{\epsilon_2}\right)^2} \qquad k_\epsilon = \sqrt{k_x^2+(\epsilon_1 k_y)^2+(\epsilon_2 k_z)^2} \qquad \epsilon_1,\epsilon_2>0$$

各向异性小波变换

定向各向异性小波

定向的各向异性GDW的二维切面

$$\epsilon_1 = 1, \epsilon_2 = 2 \quad w = 1, \ \phi = 0$$

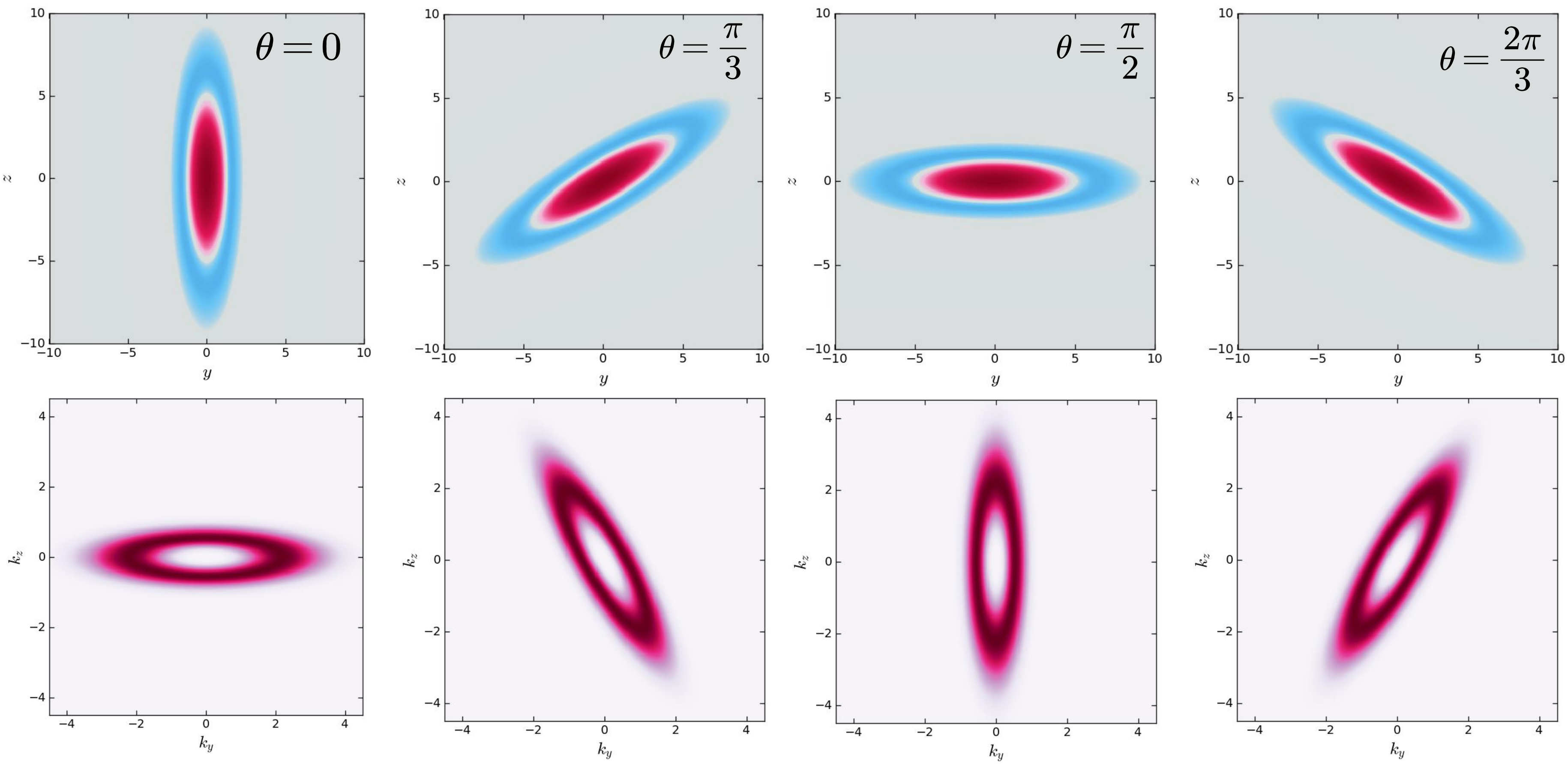


各向异性小波变换

定向各向异性小波

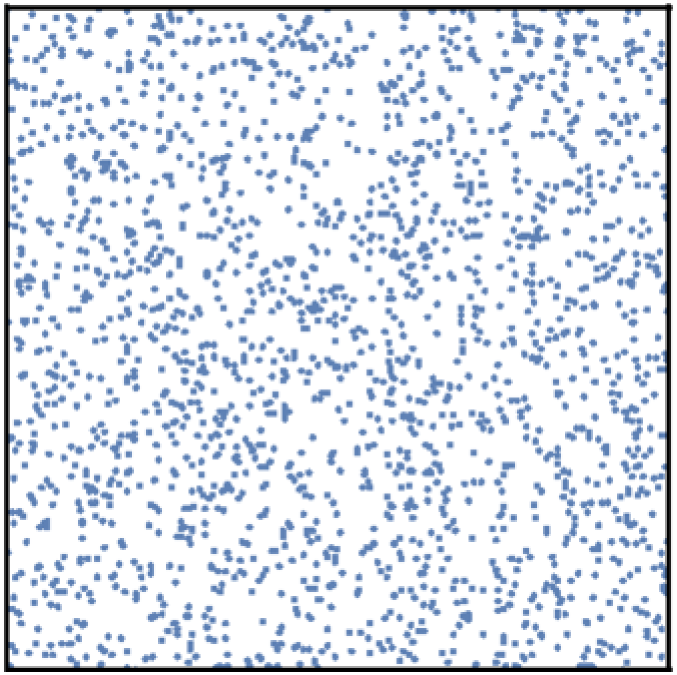
定向的各向异性CW-GDW的二维切面

$$\epsilon_1 = 1, \epsilon_2 = 2 \quad w = 1, \ \phi = 0$$

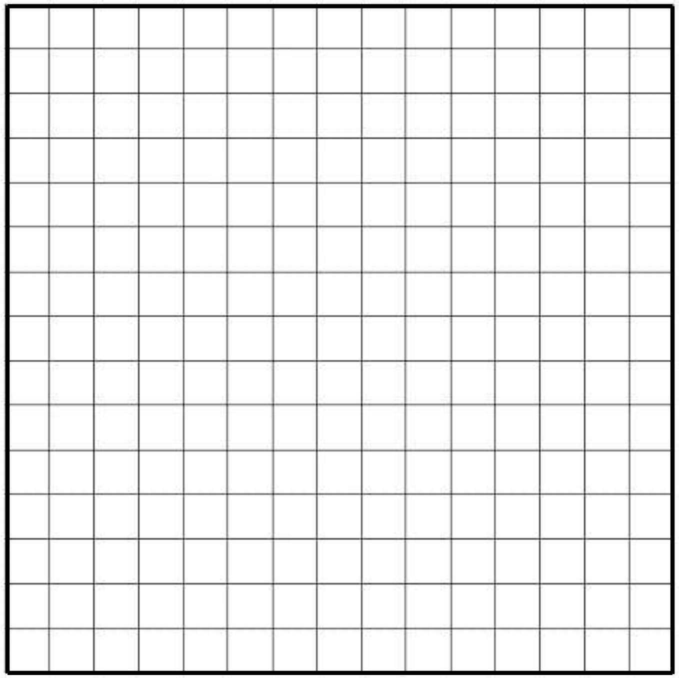


小波变换的离散化

数据类型



无规则分布的粒子数据



规则的网格场

小波变换的离散化

无规则分布的粒子数据

- 现有一边长为$V_b = L^3$的周期性立方体空间，该空间内分布着N_p 个粒子，第i个粒子的质量为m_i，则在空间$\vec{x}$处的质量密度为

$$\rho(\vec{x}) = \sum_i m_i \delta^D(\vec{x} - \vec{x}_i)$$

- 密度反差为

$$\delta(\vec{x}) = \rho(\vec{x})/\overline{\rho} - 1 = \frac{V_b}{M}\sum_i m_i \delta^D(\vec{x} - \vec{x}_i) - 1$$

- 密度反差的各向同性连续小波变换为

$$W_\delta(w,\vec{x}) = \int \delta(\vec{u})\Psi(w,\vec{x} - \vec{u})\mathrm{d}^3\vec{u}$$
$$= \frac{V_b}{M}\sum_i m_i \Psi(w,\vec{x} - \vec{x}_i)$$

- 如何对上述小波变换应用周期性边界条件?
  - 在大尺度(w值较小)和小尺度(w值较大)分别用不同方法计算

小波变换的离散化

无规则分布的粒子数据

■ 大尺度：

$$W_\delta(w,\vec{x}) = \int \delta(\vec{u})\Psi(w,\vec{x}-\vec{u})\mathrm{d}^3\vec{u}$$

$$= \sum_{m,n,q}\int_{mL}^{(m+1)L}\int_{nL}^{(n+1)L}\int_{qL}^{(q+1)L}\delta(\vec{u})\Psi(w,\vec{x}-\vec{u})\mathrm{d}u_x\mathrm{d}u_y\mathrm{d}u_z$$

令$\vec{u}' = \vec{u} - \vec{p}_{mnq}$，其中$\vec{p}_{mnq} = L(m,n,q)$

$$W_\delta(w,\vec{x}) = \sum_{m,n,q}\int_0^L\int_0^L\int_0^L\delta(\vec{u}')\Psi\left(w,\vec{x}-\vec{p}_{mnq}-\vec{u}'\right)\mathrm{d}u_x'\mathrm{d}u_y'\mathrm{d}u_z'$$

将求和号移入积分号内，并将$\vec{u}'$替换为$\vec{u}$

$$W_\delta(w,\vec{x}) = \int_{V_b}\delta(\vec{u})\Psi^P(w,\vec{x}-\vec{u})\mathrm{d}^3\vec{u}$$

$$\Psi^P(w,\vec{x}) = \sum_{m,n,q}\Psi\left(w,\vec{x}-\vec{p}_{mnq}\right)$$

周期性的小波函数

$$\delta(\vec{x}) = \frac{V_b}{M}\sum_i m_i\delta^D(\vec{x}-\vec{x}_i) - 1$$

$$W_\delta(w,\vec{x}) = \frac{V_b}{M}\sum_i m_i\Psi^P(w,\vec{x}-\vec{x}_i)$$

小波变换的离散化

无规则分布的粒子数据

- 大尺度：

$$W_\delta(w,\vec{x}) = \frac{V_b}{M}\sum_i m_i \Psi^P(w,\vec{x}-\vec{x}_i)$$

$$\Psi^P(w,\vec{x}-\vec{x}_i) = \frac{1}{V_b}\sum_{\vec{k}} \hat{\Psi}\left(w,\vec{k}\right)e^{-i\vec{k}\cdot(\vec{x}-\vec{x}_i)}$$

$$W_\delta(w,\vec{x}) = \frac{1}{M}\sum_{\vec{k}} \hat{\Psi}\left(w,\vec{k}\right)e^{-i\vec{k}\cdot\vec{x}}\sum_i m_i e^{i\vec{k}\cdot\vec{x}_i}$$

在大尺度上，$\vec{k}$的个数很少，因此上式的计算效率较高

- 小尺度：

$$W_\delta(w,\vec{x}) = \frac{V_b}{M}\sum_i m_i \Psi(w,\vec{x}-\vec{x}_i)$$

$$= \frac{V_b}{M}\sum_{|\vec{x}-\vec{x}_i|<R_\Psi(w)} m_i \Psi(w,\vec{x}-\vec{x}_i)$$

（在边界上应用周期性条件）

在小尺度上，小波在实空间有很强的局域性。因此上式中，只在小波半径内对粒子求和，极大减少了运算量。使用Tree算法可进一步提高效率

小波变换的离散化

规则的网格场

$$W_\delta(w,\vec{x}) = \int_{V_b} \delta(\vec{u})\Psi^P(w,\vec{x}-\vec{u})\mathrm{d}^3\vec{u}$$

$$\delta(\vec{x}) = \frac{1}{V_b}\sum_{\vec{k}} \hat{\delta}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$$

$$\Psi^P(w,\vec{x}) = \frac{1}{V_b}\sum_{\vec{k}} \hat{\Psi}(w,\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$$

$$W_\delta(w,\vec{x}) = \frac{1}{V_b}\sum_{\vec{k}} \hat{\delta}(\vec{k})\hat{\Psi}(w,\vec{k}) e^{-i\vec{k}\cdot\vec{x}}$$

- $\hat{\delta}(\vec{k})$可以通过FFT对网格密度场$\delta(\vec{x})$执行Fourier变换来获得
- $\hat{\Psi}(w,\vec{k})$具有明确的解析形式
- 再次借助FFT对$\hat{\delta}(\vec{k})\hat{\Psi}(w,\vec{k})$做Fourier逆变换可得$W_\delta(w,\vec{x})$

思考

- 基于一维小波，了解小波变换模极大方法(Wavelet transform modulus maxima method)，是否能将其推广到二维或三维?
- 对于各向同性小波以及可分离变量的各向异性小波，尝试推导尺度w与其对应的Fourier波数的关系；对于定向的各向异性小波，是否也存在类似的关系?
- 对于定向的各向异性小波，写出其小波变换的数学形式，思考是否存在仅对尺度积分的逆变换公式。
- 一维的Morlet小波是一个复数函数，是否能将其推广为各向同性小波、可分离变量的各向异性小波、以及定向的各向异性小波?
- 了解其他种类的各向异性小波，例如ridgelets和beamlets。

推荐阅读

- Addison, Paul S. *The illustrated wavelet transform handbook: introductory theory and applications in science, engineering, medicine and finance*. CRC press, 2017. Chapter 2

- Kaiser, Gerald, and Lonnie H. Hudgins. *A friendly guide to wavelets*. Vol. 300. Boston: Birkhäuser, 1994. Chapter 3

- Van den Berg, J. C. *Wavelets in physics*. Wavelets in Physics (2004).

- Peyrin, F., et al. *Wigner distribution and continuous wavelet transforms for image analysis: relationships and interpretation*. Proceedings of IEEE-SP International Symposium on Time-Frequency and Time-Scale Analysis. IEEE, 1994.

- https://rafat.github.io/sites/wavebook/advanced/dirwt.html